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# Basic-deformed thermostatics

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## Abstract

Starting from the *basic*-exponential, a  $q$ -deformed version of the exponential function established in the framework of the *basic*-hypergeometric series, we present a possible formulation of a generalized statistical mechanics. In a  $q$ -nonuniform lattice we introduce the *basic*-entropy related to the *basic*-exponential by means of a  $q$ -variational principle. Remarkably, this distribution exhibits a *natural* cut-off in the energy spectrum. This fact, already encountered in other formulations of generalized statistical mechanics, is expected to be relevant to the applications of the theory to those systems governed by long-range interactions. By employing the  $q$ -calculus, it is shown that the standard thermodynamic functional relationships are preserved, mimicking, in this way, the mathematical structure of the ordinary thermostatics which is recovered in the  $q \rightarrow 1$  limit.

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## 1. Introduction

There has recently been a great deal of interest in investigating deformed thermodynamic systems at the classical level. Such deformed theories are believed to deal with the statistical behaviour of complex systems, whose underlying dynamics is spanned in a multi-fractal phase space, governed by long-range interaction and/or long-time memory effects [1–4]. A possible fruitful mechanism capable of generating a deformed version of the classical statistical mechanics consists of replacing, in the Boltzmann–Gibbs distribution, the standard exponential with its deformed version, accordingly postulating a deformed entropic form which implies a generalized thermostatics theory. In this manner, some noteworthy generalizations of the standard statistical mechanics have been proposed [5–8] and their physical consequences are currently under investigation [9–11].

On the other hand, quantum algebra and quantum groups have been the subject of intensive research in several physical fields from cosmic strings and black holes to the fractional quantum Hall effect and high- $T_c$  superconductors [12]. From the seminal work of Biedenharn [13] and Macfarlane [14], it was clear that the  $q$ -calculus, originally introduced by Heine [15] and by Jackson [16] in the study of the basic-hypergeometric series [17, 18], plays a central role in the representation of the quantum groups with a deep physical meaning and not merely a mathematical exercise [19, 20]. In this context, it was shown in [21] that a natural realization of the thermostatics of  $q$ -deformed bosons and fermions can be built on the formalism of  $q$ -calculus.

Furthermore, it is remarkable to observe that the  $q$ -calculus is very well suited for to describe fractal and multi-fractal systems. As soon as the system exhibits a discrete-scale invariance, the natural tool is provided by Jackson  $q$ -derivative and  $q$ -integral, which constitute the natural generalization of the regular derivative and integral for discretely self-similar systems [22]. In fact, it was shown that  $q$ -integral is related to the free energy of spin systems on a hierarchical lattice [23].

In the recent past, some ideas of constructing a classical counterpart to the quantum group and  $q$ -deformed dynamics have been investigated [24]. Most recently, in [25], a  $q$ -deformed Poisson bracket has been developed whose underlying algebra arising from the quantum group theory appears to be invariant under the action of the  $q$ -symplectic group. This generalization implies a classical deformed dynamics and a deformed Fokker–Planck equation [26] whose stationary solution can be expressed in terms of the *basic*-exponential, the  $q$ -analogue of the exponential function in the framework of the basic-hypergeometric series (henceforward, we will use the term *basic*-exponential and *basic*-thermostatics to avoid any confusion with the  $q$ -exponential customarily employed in the Tsallis' thermostatics formulation [5]).

The previous investigations raise the interesting question whether the  $q$ -calculus and the *basic*-exponential can be introduced, as a starting point, for the study of a deformed statistical thermodynamics at the classical level. Just as the quantum  $q$ -deformation plays a crucial role in the interpretation of several complex physical systems, we expect that a classical  $q$ -deformation of the thermostatics can be relevant in several physical applications. It is thus worthwhile to investigate the structure of a classical statistical mechanics where the probability distribution function is given by employing the *basic*-exponential. This investigation represents the primary goal of this paper.

Relating to the previously quoted generalized statistical theories existing in the literature [5–8], it must be stressed that the deformed exponential functions are very different from that we are introducing in the present work. The difference arises mainly in the asymptotic behaviour of the *basic*-exponential which fails to show a power-law tail. Nevertheless, some mathematical peculiarities exhibited by the *basic*-exponential make it relevant in the construction of a generalized statistical mechanics. First among them is the existence of a *natural* cut-off in the energy (velocity) spectrum which appears to be closely related to the presence of long-range interactions among constituents of the system. To the best of our knowledge, this was first suggested in [27] in order to overcome the infinity arising in the mass and radius of an isothermal globular cluster. More recently, concerning several physical systems, there have been many investigations [28–35] on the relevance of the presence of an energy cut-off behaviour in the particle distribution functions.

Finally, we should stress that the *basic*-exponential introduced in the present work is also employed in the formulation of other interesting generalizations of physical theories such as, for instance, in the  $q$ -deformed Schrödinger theory describing the deformed quantum harmonic oscillator [36] or in the  $q$ -deformed theory of quantum coherence of bosons [37, 38].

Our paper is organized as follows. In section 2, for convenience of the reader, we review the main mathematical properties concerning the quantum algebra of real numbers. We introduce the *basic*-exponential by means of a power series and derive its main properties which will be used in the formulation of the theory. In section 3, we introduce the *basic*-entropy and, by means of a  $q$ -version of the variational principle, we determine the deformed distribution, for each case in turn, of a microcanonical, canonical and grand canonical system. The thermodynamic structure of the theory is explored in section 4, while in section 5 we apply the present formalism to a system of  $q$ -interacting particles whose  $q \rightarrow 1$  limit reduces to the free system. Conclusions are reported in section 6 and we end the paper with two mathematical appendices.

## 2. Mathematical background

We shall begin by recalling the main features of the  $q$ -calculus for real numbers. It is based on the following  $q$ -commutative relation among the operators  $\hat{x}$  and  $\hat{\partial}_x$ ,

$$\hat{\partial}_x \hat{x} = 1 + q \hat{x} \hat{\partial}_x, \quad (2.1)$$

with  $q$  a real and positive parameter.

A realization of the above algebra in terms of ordinary real numbers can be accomplished by the replacement [39]

$$\hat{x} \rightarrow x, \quad (2.2)$$

$$\hat{\partial}_x \rightarrow \mathcal{D}_x, \quad (2.3)$$

where  $\mathcal{D}_x$  is the Jackson derivative [16] defined as

$$\mathcal{D}_x = \frac{q^{x\hat{\partial}_x} - 1}{(q-1)x}. \quad (2.4)$$

Its action on an arbitrary real function  $f(x)$  is given by

$$\mathcal{D}_x f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \quad (2.5)$$

The Jackson derivative satisfies some simple properties which will be useful in the following. For instance, its action on a monomial  $f(x) = x^n$  is given by

$$\mathcal{D}_x x^n = [n]x^{n-1}, \quad (2.6)$$

and

$$\mathcal{D}_x x^{-n} = -\frac{[n]}{q^n} \frac{1}{x^{n+1}}, \quad (2.7)$$

where  $n \geq 0$ , and

$$[n] = \frac{q^n - 1}{q - 1} \quad (2.8)$$

are *basic*-numbers. By linearity, we can extend the action of Jackson derivative to a generic polynomial. Moreover, we can easily verify the following  $q$ -version of the Leibnitz rule:

$$\begin{aligned} \mathcal{D}_x(f(x)g(x)) &= \mathcal{D}_x f(x)g(x) + f(qx)\mathcal{D}_x g(x), \\ &= \mathcal{D}_x f(x)g(qx) + f(x)\mathcal{D}_x g(x). \end{aligned} \quad (2.9)$$

A relevant role in the  $q$ -algebra, as developed by Jackson, is given by the *basic*-binomial series defined by

$$\begin{aligned}(x+y)^{(n)} &= (x+y)(x+qy)(x+q^2y)\cdots(x+q^{n-1}y) \\ &\equiv \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2} x^{n-r} y^r,\end{aligned}\quad (2.10)$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}\quad (2.11)$$

is known as the  $q$ -binomial coefficient which reduces to the ordinary binomial coefficient in the  $q \rightarrow 1$  limit [18]. We should remark that equation (2.11) holds for  $0 \leq r \leq n$ , while it is assumed to vanish otherwise and we have defined  $[n]! = [n][n-1]\cdots[1]$ . Equation (2.10) can be easily generalized to an arbitrary polynomial as shown in appendix B.

Remarkably, a  $q$ -analogue of the Taylor expansion has been introduced in [16] by means of a *basic*-binomial (2.10) as

$$f(x) = f(a) + \frac{(x-a)^{(1)}}{[1]!} \mathcal{D}_x f(x) \Big|_{x=a} + \frac{(x-a)^{(2)}}{[2]!} \mathcal{D}_x^2 f(x) \Big|_{x=a} + \cdots,\quad (2.12)$$

where  $\mathcal{D}_x^2 \equiv \mathcal{D}_x \mathcal{D}_x$  and so on.

Consistently with the  $q$ -calculus, we also introduce the *basic*-integration

$$\int_0^{\lambda_0} f(x) d_q x = \sum_{n=0}^{\infty} \Delta_q \lambda_n f(\lambda_n),\quad (2.13)$$

where  $\Delta_q \lambda_n = \lambda_n - \lambda_{n+1}$  and  $\lambda_n = \lambda_0 q^n$  for  $0 < q < 1$  whilst  $\Delta_q \lambda_n = \lambda_{n-1} - \lambda_n$  and  $\lambda_n = \lambda_0 q^{-n-1}$  for  $q > 1$  [17, 18, 22]. Clearly, equation (2.13) is reminiscent of the Riemann quadrature formula performed now in a  $q$ -nonuniform hierarchical lattice with a variable step  $\Delta_q \lambda_n$ . It is trivial to verify that

$$\mathcal{D}_x \int_0^x f(y) d_q y = f(x),\quad (2.14)$$

for any  $q > 0$ .

Let us now introduce the following  $q$ -deformed function defined by the series

$$E_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = 1 + x + \frac{x^2}{[2]!} + \frac{x^3}{[3]!} + \cdots,\quad (2.15)$$

which will play the main role in the framework we are introducing. The function (2.15) defines the *basic*-exponential, well known in the literature since a long time ago, originally introduced in the study of basic-hypergeometric series [15, 17, 18]. In this context, let us observe that definition (2.15) is fully consistent with its Taylor expansion, as given by equation (2.12).

In the following, we briefly review the main algebraic properties of the *basic*-exponential useful for our developments.

The *basic*-exponential is a monotonically increasing function,  $dE_q(x)/dx > 0$ , convex,  $d^2E_q(x)/dx^2 > 0$ , with  $E_q(0) = 1$  and reducing to the ordinary exponential in the  $q \rightarrow 1$  limit:  $E_1(x) \equiv \exp(x)$ .

An important property satisfied by the  $q$ -exponential can be written formally as [18]

$$E_q(x+y) = E_q(x)E_{1/q}(y),\quad (2.16)$$

where the left-hand side of equation (2.16) must be considered by means of its series expansion in terms of *basic*-binomials:

$$E_q(x + y) = \sum_{k=0}^{\infty} \frac{(x + y)^{(k)}}{[k]!}. \tag{2.17}$$

By observing that  $(x - x)^{(k)} = 0$  for any  $k > 0$ , since  $(x - x)^{(0)} = 1$ , from equation (2.16) we obtain

$$E_q(x)E_{1/q}(-x) = 1. \tag{2.18}$$

From the above relations, we easily deduce that, if  $q < 1$ , the series (2.15) converges for all finite values of  $x < 1/(1 - q)$ , otherwise, if  $q > 1$  the series converges for  $x > q/(1 - q)$ .

Thus, we can summarize the asymptotic behaviour of the *basic*-exponential as

$$E_q(-\infty) = 0, \quad E_q\left(\frac{1}{1 - q}\right) = +\infty, \quad \text{if } q < 1, \tag{2.19}$$

$$E_q\left(\frac{q}{1 - q}\right) = 0, \quad E_q(+\infty) = +\infty, \quad \text{if } q > 1, \tag{2.20}$$

where it is important to mention that the first expression of equation (2.20) defines a cut-off condition of the *basic*-exponential in the region relevant for the following developments.

Among many properties, it is important to recall the following relation [18]:

$$\mathcal{D}_x E_q(ax) = aE_q(ax), \tag{2.21}$$

and its dual

$$\int_0^x E_q(ay) \, d_q y = \frac{1}{a}[E_q(ax) - 1]. \tag{2.22}$$

It must be pointed out that equations (2.21) and (2.22) are two important properties of the *basic*-exponential which turns out to be not true if we employ the ordinary derivative or integral. In particular, from equation (2.21) for  $a = 1$ , we can obtain the further useful relation

$$E_q(qx) = [1 + (q - 1)x]E_q(x). \tag{2.23}$$

Moreover, from equation (2.22) with  $a < 0$ , we have

$$\int_0^{x_{\max}} E_q(ay) \, d_q y = -\frac{1}{a}, \tag{2.24}$$

where  $x_{\max} \rightarrow +\infty$  in the  $q > 1$  case, while  $x_{\max} = 1/(1 - q)$  in the  $q < 1$  case, accounting for the cut-off condition (2.19).

In addition to the *basic*-exponential, we can also introduce its inverse function, the *basic*-logarithm  $\text{Ln}_q(x)$ , such that

$$E_q(\text{Ln}_q(x)) = \text{Ln}_q(E_q(x)) = x, \tag{2.25}$$

which certainly exists because  $E_q(x)$  is a strictly monotonic function. Many properties of the *basic*-logarithm follow directly from the corresponding ones of the *basic*-exponential. For instance,  $\text{Ln}_q(x)$  is a monotonic, increasing and concave function ( $d \text{Ln}_q(x)/dx > 0$ ,  $d^2 \text{Ln}_q(x)/dx^2 < 0$ ), normalized in  $\text{Ln}_q(1) = 0$  and the asymptotic behaviour is given by

$$\text{Ln}_q(0) = -\infty, \quad \text{Ln}_q(+\infty) = \frac{1}{1 - q}, \quad \text{if } q < 1, \tag{2.26}$$

$$\text{Ln}_q(0) = \frac{q}{1 - q}, \quad \text{Ln}_q(+\infty) = +\infty, \quad \text{if } q > 1. \tag{2.27}$$

Although a definition of  $\text{Ln}_q(x)$  through a series is possible, it appears to be a nontrivial task to write it in an easy form and, to the best of our knowledge, there are no definitive results in the literature (see for instance [41]).

We conclude this section by remarking that alternative definitions of the *basic*-exponential, by means of a different definition of *basic*-numbers, have been widely employed in the literature [42]. Among the many, we may point out the choice based on the symmetric definition  $[n]_s = (q^n - q^{-n})/(q - q^{-1})$ . The corresponding symmetric *basic*-exponential may be defined on the whole real region  $(-\infty, +\infty)$  and has the symmetry  $q \rightarrow 1/q$ . As a consequence, the symmetric *basic*-exponential does not present the cut-off feature.

### 3. Basic-entropy and its distribution

Equipped with the *basic*-functions, our aim is to formulate a statistical mechanics based on the formalism of the  $q$ -algebra and to study its main physical implications.

On the basis of the above mathematical framework, it appears natural to generalize the Boltzmann entropy to the following form:

$$S_q(p) = - \int_{\mathcal{M}} p(\lambda) \text{Ln}_q p(\lambda) d_q \lambda, \quad (3.1)$$

where  $p(\lambda)$  is probability distribution function labelled by a set of parameters  $\lambda$  running on the manifold  $\mathcal{M}$ , eventually identified with the phase space, which define the accessible states of the system. Henceforward we adopt units where the Boltzmann constant  $k_B = 1$ .

Equation (3.1) resembles the well-known Boltzmann–Gibbs entropy  $S^{\text{BG}}(p)$  through the replacement of the logarithm with the *basic*-logarithm. Clearly, the function  $S_q(p)$  reduces to the standard entropy  $S^{\text{BG}}(p) \equiv S_1(p)$  in the  $q \rightarrow 1$  limit. In the following, we shall refer to the function (3.1) as *basic*-entropy.

A way to obtain the equilibrium distribution from the entropy  $S_q(p)$ , given a set of  $M + 1$  constraints  $\Phi_j(p)$  with  $j = 0, \dots, M$ , can be accomplished through the following variational problem:

$$\delta \mathcal{F}(p) = 0, \quad (3.2)$$

with

$$\mathcal{F}(p) = \left( S_q(p) - \sum_j \mu_j^* \Phi_j(p) \right), \quad (3.3)$$

where  $\mu_j^*$  are the Lagrangian multipliers associated with the constraints  $\Phi_j(p)$ .

Quite generally, such constraints can be written as

$$\Phi_j(p) = \int_{\mathcal{M}} \phi_j(\lambda) p(\lambda) d_q \lambda, \quad (3.4)$$

representing the mean value of the quantities  $\phi_j(\lambda)$  which are identifiable with a suitable physical observable. In particular, for  $\phi_0(\lambda) = 1$ , equation (3.4) gives the normalization of the distribution function.

In order to easily handle the variation problem of equation (3.2), let us introduce the ansatz

$$p(\lambda) = E_q(-f(\lambda)). \quad (3.5)$$

We recall that  $E_q(-x)$  is a strictly monotonically decreasing function. This means that it reaches its minimum at points which maximize the functions  $f(\lambda)$ .

Actually, the problem, equation (3.2), can be replaced by the following equivalent  $q$ -variational problem:

$$\delta_q \tilde{\mathcal{F}}(f) = \mathcal{D}_f \tilde{\mathcal{F}}(f) \Delta_q f = 0, \tag{3.6}$$

with  $\Delta_q f = (q - 1)f$  and

$$\tilde{\mathcal{F}}(f) \equiv \mathcal{F}(p(f)) = \int_{\mathcal{M}} E_q(-f(\lambda)) \left( f(\lambda) - \sum_j \mu_j \phi_j(\lambda) \right) d_q \lambda. \tag{3.7}$$

It is shown in appendix A that, according to the  $q$ -algebra, both the calculus (3.2) and (3.6) give substantially the same result (apart from a redefinition of the Lagrange multipliers,  $\mu_j \rightarrow \mu_j^*$ , which, accounting for the relevant constraint equations, has no effect on the expression of the final distribution). Equation (3.6) can be solved more speedily as

$$\begin{aligned} \delta_q \tilde{\mathcal{F}}(p) &= \left[ \mathcal{D}_f \int_{\mathcal{M}} E_q(-f(\lambda)) \left( f(\lambda) - \sum_j \mu_j \phi_j(\lambda) \right) d_q \lambda \right] \Delta_q f \\ &= \left[ \int_{\mathcal{M}} E_q(-f(\lambda)) \left( qf(\lambda) - 1 - \sum_j \mu_j \phi_j(\lambda) \right) d_q \lambda \right] \Delta_q f = 0, \end{aligned} \tag{3.8}$$

which implies the following relation:

$$f(\lambda) = q^{-1} \left( 1 + \sum_j \mu_j \phi_j(\lambda) \right), \tag{3.9}$$

and accounting for equation (3.5) we obtain the general solution in the form

$$p(\lambda) = E_q \left( -q^{-1} \left( 1 + \sum_j \mu_j \phi_j(\lambda) \right) \right). \tag{3.10}$$

It is easy to verify that this expression reduces, in the  $q \rightarrow 1$  limit, to the ordinary Gibbs distribution. Let us separately consider three main cases: the microcanonical system, the canonical system and the grand canonical system.

### 3.1. Microcanonical system

We consider a closed system with a given fixed energy  $U$ , volume  $V$  and particle number  $N$ . In this case, the system is forced by the only constraint

$$\int_{\mathcal{M}} p(\lambda) d_q \lambda = 1, \tag{3.1.1}$$

which assures the normalization of the distribution.

Before proceeding, let us spend a few word about equation (3.1.1) to better understand the underlying physical meaning of the  $q$ -calculus. By taking into account the definition of the *basic*-integral (2.13), for  $0 < q < 1$ , we obtain

$$\int_{\mathcal{M}} p(\lambda) d_q \lambda = \lambda_0(1 - q)[p(\lambda_0) + qp(\lambda_1) + q^2p(\lambda_2) + \dots], \tag{3.1.2}$$

where  $\lambda_0$  is a constant and  $\lambda_i = q^i \lambda_0$ . This expression shows two important features of the theory we are developing. First, the  $q$ -integral plunges, in a natural way, to consider a  $q$ -deformed lattice whose amplitude of the elementary cell  $\Delta_q \lambda_0 = \lambda_0(1 - q)$  is shrunken,



step by step, by the quantity  $q^n$ . This is a substantially different situation with respect to the standard case obtained in the  $q \rightarrow 1$  limit where any probability (differential probability) is multiplied by an equal (infinitesimal) quantity:  $dp(x) = p(x) dx$ . Second, the parameter space assumes a fractal structure given by the rule  $\lambda_n = q^n \lambda_0$ . We recognize a self-similarity in the parameter space since, starting from any level  $N > 1$ , the same structure  $\lambda_{N+n} = q^n \lambda_N$  is discovered.

Remarkably, by introducing a set of discrete probability distributions  $p_n$ , related to  $p(\lambda)$ , as

$$p_n = \Delta_q \lambda_n p(\lambda_n), \quad (3.1.3)$$

equation (3.1.1) becomes

$$\sum_{i=0}^{\infty} p_i = 1. \quad (3.1.4)$$

Clearly, the same considerations also hold in the  $q > 1$  case.

In order to derive the expression for  $p(\lambda)$  we introduce the following constrained entropic functional:

$$\mathcal{F}(p) = - \int_{\mathcal{M}} p(\lambda) [\text{Ln}_q p(\lambda) + \alpha] d_q \lambda, \quad (3.1.5)$$

where  $\alpha$  is the Lagrange multiplier associated with equation (3.1.1).

By inserting the ansatz (3.5) into equation (3.1.5) we obtain

$$\tilde{\mathcal{F}}(f) = \int_{\mathcal{M}} E_q(-f(\lambda))(f(\lambda) - \alpha) d_q \lambda. \quad (3.1.6)$$

By evaluating the equation  $\delta_q \tilde{\mathcal{F}}(f) = 0$ , accounting for equations (2.9) and (2.21), we obtain

$$\delta_q \tilde{\mathcal{F}}(f) = E_q(-f(\lambda))(1 + \alpha - qf(\lambda)) = 0. \quad (3.1.7)$$

The above expression implies the following equation:

$$\text{Ln}_q p(\lambda) + q^{-1}(1 + \alpha) = 0, \quad (3.1.8)$$

which gives the microcanonical distribution in the form

$$p(\lambda) = E_q(-q^{-1}(1 + \alpha)). \quad (3.1.9)$$

(We refer to appendix B for a systematic derivation of equation (3.1.9)).

Since the above expression does not depend on  $\lambda$ , by accounting for the condition (3.1.1), we obtain the expected microcanonical uniform distribution

$$p = \frac{1}{W}, \quad (3.1.10)$$

where the number of accessible states

$$W(U, V, N) = \int_{\mathcal{M}} d_q \lambda \quad (3.1.11)$$

is related to the Lagrange multiplier through the relation  $\alpha = -1 - q \text{Ln}_q(1/W)$ , which is a function of the energy, the volume and the total number of particles in the system.

### 3.2. Canonical system

By following the steps described in the previous subsection, we can derive the canonical distribution for an open system that exchanges energy with the surrounding. In this case, let us pose  $\epsilon \equiv \epsilon(\lambda)$  the energy of the system still labelled through  $\lambda$  so that, along with the constraint (3.1.1), we impose the further condition on the mean energy

$$\int_{\mathcal{M}} \epsilon(\lambda) p(\epsilon(\lambda)) d_q \lambda = \langle \epsilon \rangle. \quad (3.2.1)$$

Accounting for definition (2.13) this last condition becomes ( $0 < q < 1$ )

$$\lambda_0(1-q)[\epsilon_0 p(\epsilon_0) + q\epsilon_1 p(\epsilon_1) + q^2\epsilon_2 p(\epsilon_2) + \dots] = \langle \epsilon \rangle, \quad (3.2.2)$$

where  $\epsilon_i \equiv \epsilon(\lambda_i)$  so that the  $q$ -calculus implies a fractal structure in the energy spectrum. With the position (3.1.3) we realize that the constraint (3.2.1) on the mean energy is equivalent to the standard definition

$$\sum_{i=0}^{\infty} \epsilon_i p_i = \langle \epsilon \rangle, \quad (3.2.3)$$

although the fractal structure in the energy spectrum still holds being implicitly contained in the definition of the discrete probabilities  $p_i$ .

After introducing the constrained entropic functional

$$\mathcal{F}(p) = - \int_{\mathcal{M}} p(\epsilon(\lambda)) [\text{Ln}_q p(\epsilon(\lambda)) + \alpha + \beta \epsilon(\lambda)] d_q \lambda, \quad (3.2.4)$$

with  $\alpha$  and  $\beta$  being the Lagrange multiplier associated with equations (3.1.1) and (3.2.1), by imposing the ansatz (3.5) we deal with the variational problem  $\delta_q \tilde{\mathcal{F}}(f) = 0$  where

$$\tilde{\mathcal{F}}(f) = \int_{\mathcal{M}} E_q(-f(\epsilon(\lambda))) [f(\epsilon(\lambda)) - \alpha - \beta \epsilon(\lambda)] d_q \lambda, \quad (3.2.5)$$

and whose solution reads

$$\text{Ln}_q p(\epsilon) + q^{-1}(1 + \alpha + \beta \epsilon) = 0. \quad (3.2.6)$$

According to the  $q$ -algebra described in equation (B.4), with  $x = \text{Ln}_q p(\epsilon)$ ,  $y = q^{-1}(1 + \alpha)$  and  $z = q^{-1}\beta\epsilon$ , from equation (3.2.6) we derive the following canonical distribution:

$$p(\epsilon) = E_q(-q^{-1}(1 + \alpha)) E_q(-\beta_q \epsilon), \quad (3.2.7)$$

where  $\beta_q = q^{-1}\beta$  (see appendix B). By imposing the normalization condition (3.1.1) on the distribution  $p(\epsilon)$ , we obtain

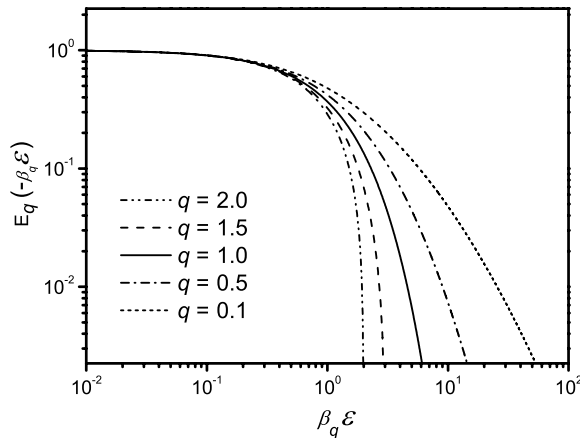
$$p(\epsilon) = \frac{1}{Z_q} E_q(-\beta_q \epsilon), \quad (3.2.8)$$

where  $Z_q$  is the canonical partition function defined by

$$Z_q = E_{1/q}(q^{-1}(1 + \alpha)) = \int_{\mathcal{M}} E_q(-\beta_q \epsilon(\lambda)) d_q \lambda. \quad (3.2.9)$$

Trivially, equation (3.2.8) reduces to the canonical Gibbs distribution in the  $q \rightarrow 1$  limit.

We remark that, for  $q < 1$  all the energy levels  $\epsilon$  can be occupied. For  $q > 1$ , however, the distribution (3.2.8) shows a cut-off in the energy spectrum due to the finite convergence radius of the function  $E_q(x)$ . This is an important consequence of the theory under investigation which limits the number of states accessible to the system. Its origin can be related, as generally accepted, to the presence of interactions among the parts of the system, whose effect



**Figure 1.** Generalized Boltzmann distribution  $E_q(-\beta_q \epsilon)$  for different values of the deformed parameter  $q$ . The solid line coincides with the standard exponential function  $\exp(-\beta \epsilon)$ .

is to reduce the volume of the phase space. This peculiarity is also encountered in other statistics mechanical models based on generalized entropic forms.

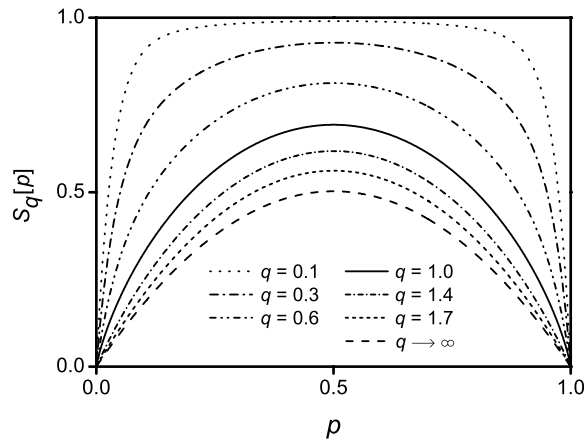
In particular, equation (2.20) imposes the following limiting condition on the energy levels  $\epsilon < \epsilon_{\max}$ , where

$$\epsilon_{\max} = \frac{q}{(q-1)\beta_q}. \quad (3.2.10)$$

Physically, this means that all the microscopic configurations of the phase space corresponding to an energy  $\epsilon$  beyond  $\epsilon_{\max}$  are statistically unattainable. We remark that  $\epsilon_{\max}$  is a function of  $\beta_q$  which plays the role of the inverse of a pseudo temperature. Its value is determined through equation (3.2.1) and it is expected, like in the undeformed theory, that  $\beta_q$  decreases as the mean energy increases. It means that the cut-off condition plays a relevant role in those *small* systems whose mean energy is small compared with the typical energy values of the macroscopic systems. This is reminiscent of a quantum scenario although our system is a classical one. In figure 1, we illustrate the behaviour of the basic Boltzmann factor  $E_q(-\beta_q \epsilon)$  for different values of the deformation parameter  $q$  compared with the classical one given by  $q = 1$ . It is observed that for  $q < 1$  high energy events are enhanced with respect to the standard case while, for  $q > 1$ , events are more and more inhibited when energy increases until it reaches the cut-off point, where  $p(\epsilon) = 0$ .

In figure 2, we show the *basic*-entropy for a system of two levels with probabilities  $p$  and  $1 - p$ , respectively, for different values of the deformation parameter. The dashed line represents the asymptotic curve reached for  $q \gg 1$ .

Again, we stress that such a cut-off feature is also shown by other distributions obtained from physically motivated generalizations of the Boltzmann–Gibbs entropy. For instance, it can be shown [10] that for suitably chosen deformation parameters generalized entropies belonging to the two-parameter family of Sharma–Mittal [43], which also include among the others the well-known Tsallis’ entropy [5] and the R enyi’s entropy [44], generate probability distribution functions which exhibit a cut-off in their tails. Nevertheless, it is worthwhile to observe that almost all the members belonging to the Sharma–Mittal family, with some exception such as the Boltzmann–Gibbs entropy and the Gaussian entropy, have an asymptotic power-law behaviour. This differs substantially from the asymptotic behaviour shown by *basic*-distribution, which is more similar to that of the stretched exponential. In this respect,



**Figure 2.** Plot of *basic-entropy*, for different values of the deformation parameter, for a system with two levels. The dashed line represents the asymptotic curve reached for  $q \gg 1$ .

the theory under investigation is not an alternative but complementary to the already existing generalized version of the statistical mechanics, since it can be relevant in the study of those complex systems which are not characterized by an asymptotic free-scale behaviour.

3.3. Grand canonical system

Finally, let us investigate the grand canonical distribution describing an open system where energy and particles can be exchanged with the surrounding. This can be accomplished by imposing the following constraints

$$\sum_N \int_{\mathcal{M}} p_N(\epsilon(\lambda)) d_q \lambda = 1, \tag{3.3.1}$$

$$\sum_N \int_{\mathcal{M}} \epsilon(\lambda) p_N(\epsilon(\lambda)) d_q \lambda = \langle \epsilon \rangle, \tag{3.3.2}$$

$$\sum_N \int_{\mathcal{M}} N p_N(\epsilon(\lambda)) d_q \lambda = \langle N \rangle, \tag{3.3.3}$$

on the normalization, the mean energy and the mean particle number, where  $N = 1, \dots, \infty$ , enumerate the particles contained in the system.

The distribution function is found just as in the former cases. We form the constrained entropic functional

$$\mathcal{F}(p) = - \sum_N \int_{\mathcal{M}} p_N(\epsilon(\lambda)) [\text{Ln}_q p_N(\epsilon(\lambda)) + \alpha + \beta \epsilon(\lambda) + \gamma N] d_q \lambda, \tag{3.3.4}$$

where  $\alpha, \beta$  and  $\gamma$  are the Lagrange multipliers associated with the constraints (3.3.1)–(3.3.3), respectively. By evaluating the equation  $\delta_q \tilde{\mathcal{F}}(f) = 0$ , with

$$\tilde{\mathcal{F}}(f) = \sum_N \int_{\mathcal{M}} E_q(-f_N(\epsilon(\lambda))) [f_N(\epsilon(\lambda)) - \alpha - \beta \epsilon(\lambda) - \gamma N] d_q \lambda, \tag{3.3.5}$$

obtained from equation (3.3.4) by using the ansatz  $p_N(\epsilon) = E_q(-f_N(\epsilon))$ , we derive the following result:

$$\text{Ln}_q p_N(\epsilon) + q^{-1}(1 + \alpha + \beta \epsilon - \mu \beta N) = 0, \tag{3.3.6}$$

where we set  $\gamma = -\mu\beta$ . According to the  $q$ -algebra (B.4) with  $x = \text{Ln}_q p_N(\epsilon)$ ,  $y = q^{-1}(1 + \alpha)$ ,  $z = q^{-1}\beta\epsilon$  and  $u = -q^{-1}\mu\beta N$ , from equation (3.3.6) we obtain the grand canonical distribution function in the form

$$p_N(\epsilon) = \frac{1}{Z_q} E_q(-\beta_q \epsilon) E_q(\mu \beta_q N), \quad (3.3.7)$$

where

$$Z_q = E_{1/q}(q^{-1}(1 + \alpha)) = \sum_N \int_{\mathcal{M}} E_q(-\beta_q \epsilon(\lambda)) E_q(\mu \beta_q N) d_q \lambda \quad (3.3.8)$$

is the grand partition function.

#### 4. Some thermodynamic relations

In the following, we shall investigate the thermodynamic structure of the theory. It is shown that some basic relations of the standard theory can be transcribed in a straightforward manner in the present formalism establishing, in this way, the ground for a generalized classical thermodynamics based on the framework of the  $q$ -algebra.

To begin with, we observe that by multiplying equation (3.3.6) by  $p_N(\epsilon)$  and taking into account all the constraints imposed on the system, we obtain the relation

$$S_q(\langle \epsilon \rangle, \langle N \rangle, V) = \text{Ln}_{1/q} Z_q + \beta_q \langle \epsilon \rangle - \mu \beta_q \langle N \rangle, \quad (4.1)$$

where we set  $\text{Ln}_{1/q} Z_q = q^{-1}(1 + \alpha)$  as it follows from equation (3.3.8). Equation (4.1) mimics the standard relation  $S = \ln Z + \beta U - \mu \beta N$  which is recovered in the  $q \rightarrow 1$  limit.

If one is willing to identify the quantity  $\beta_q$  with the inverse of the temperature  $\beta_q = 1/T$ , equation (4.1) can be rewritten as

$$\langle \epsilon \rangle = T S_q - T \text{Ln}_{1/q} Z_q + \mu \langle N \rangle. \quad (4.2)$$

On the other hand, in analogy with standard thermodynamics, we can introduce the pressure  $P$  and the volume  $V$  of the system by requiring that all the thermodynamics variables are functionally related through the following relationship:

$$\langle \epsilon \rangle = T S_q - P V + \mu \langle N \rangle, \quad (4.3)$$

which is a constitutive equation for the theory under investigation and can be identified with the  $q$ -analogue of the Euler's equation [45]. By comparing equations (4.2) and (4.3) we are thus encouraged to define the pressure through the relation

$$P V = T \text{Ln}_{1/q} Z_q, \quad (4.4)$$

which represents the  $q$ -generalized state equation for a system described by the *basic*-entropy.

The partition function (3.2.9), as well as the grand partition function (3.3.8), is a useful tool to evaluate the statistical proprieties of the system. In fact, by evaluating the Jackson derivative of  $Z_q$  as

$$\mathcal{D}_{\beta_q} Z_q = \mathcal{D}_{\beta_q} \int_{\mathcal{M}} E_q(-\beta_q \epsilon(\lambda)) d_q \lambda = - \int_{\mathcal{M}} \epsilon(\lambda) E_q(-\beta_q \epsilon(\lambda)) d_q \lambda, \quad (4.5)$$

where equation (2.21) has been used, we obtain the result

$$\langle \epsilon \rangle = - \frac{1}{Z_q} \mathcal{D}_{\beta_q} Z_q, \quad (4.6)$$

which in the  $q \rightarrow 1$  limit reduces to the well-known relation  $\langle \epsilon \rangle = -(dZ/d\beta)/Z$ .

Similar results can be obtained starting from the function  $Z_q$  as follows:

$$\langle \epsilon \rangle = -\frac{1}{Z_q} \mathcal{D}_{\beta_q} Z_q, \quad (4.7)$$

$$\langle N \rangle = \frac{1}{\beta_q Z_q} \mathcal{D}_\mu Z_q. \quad (4.8)$$

Incidentally, by using the prescription

$$\epsilon \rightarrow \epsilon + \delta A(\epsilon), \quad (4.9)$$

where  $A(\epsilon)$  is an arbitrary physical observable, the expectation value of  $A$  can be obtained as

$$\langle A \rangle = -\frac{1}{\beta_q Z_q} \mathcal{D}_\delta Z_q \Big|_{\delta=0}. \quad (4.10)$$

In this sense, the partition function encodes all the statistical information contained in the system.

Another physically relevant quantity is obtained starting from the expression

$$\mathcal{D}_{\beta_q} \langle \epsilon \rangle = \mathcal{D}_{\beta_q} \left( \frac{1}{Z_q(\beta_q)} \int_{\mathcal{M}} \epsilon(\lambda) E_q(-\beta_q \epsilon(\lambda)) d_q \lambda \right), \quad (4.11)$$

where we have explicitly indicated the dependence on  $\beta_q$  in the partition function. By taking into account equation (2.23), we can derive the result

$$\mathcal{D}_{\beta_q} \langle \epsilon \rangle = -\frac{\langle \epsilon^2 \rangle - \langle \epsilon \rangle^2}{1 - (q-1)\beta_q \langle \epsilon \rangle} \equiv -\sigma_{q,\epsilon}^2, \quad (4.12)$$

where  $\langle \epsilon^2 \rangle = \int \epsilon^2 p(\epsilon) d_q \lambda$ .

In the  $q \rightarrow 1$  limit, we obtain the classical relation  $d\langle \epsilon \rangle / d\beta = -\sigma_\epsilon^2$ .

The quantity  $\sigma_{q,\epsilon}$  measures the fluctuation of the energy of the system around its mean value  $\langle \epsilon \rangle$ . It is observed that, compared to the classical case, such fluctuations are reduced for  $q < 1$  and are enlarged for  $q > 1$ .

Alternatively, by mimicking the classical definition, one can introduce the heat capacity by

$$C_V = -\beta_q^2 \mathcal{D}_{\beta_q} \langle \epsilon \rangle \equiv (\beta_q \sigma_{q,\epsilon})^2, \quad (4.13)$$

so that the relative width of the fluctuations in energy is given by

$$\frac{\sigma_{q,\epsilon}}{\langle \epsilon \rangle} = \frac{1}{\beta_q \langle \epsilon \rangle} \sqrt{C_V}. \quad (4.14)$$

Similar derivations can also be made for the grand canonical case where, together with equation (4.12), we can also obtain the further relation

$$\mathcal{D}_\mu \langle N \rangle = \beta_q \frac{\langle N^2 \rangle - \langle N \rangle^2}{1 + (q-1)\mu \beta_q \langle N \rangle} \equiv \sigma_{q,N}^2, \quad (4.15)$$

with  $\langle N^2 \rangle = \sum_N \int N^2 p_N(\epsilon) d_q \lambda$ , which measures the deviation from the mean value  $\langle N \rangle$ .

## 5. Basic-ideal gas

In this section, to illustrate the consequences of our generalized thermodynamical model, we discuss a simple *basic*-noninteracting particle gas which reduces, in the  $q \rightarrow 1$  limit, to the well-known ideal gas.

Let us start from the following Hamiltonian  $H_0$  describing a system of  $N$  identical particles:

$$H_0(\vec{p}) = \sum_{i=1}^N \frac{p_i^2}{2m}, \quad (5.1)$$

where  $\vec{p} \equiv (p_1, \dots, p_N)$  is the  $3N$ -vector momenta. In order to be consistent with our model, we require that the momenta  $p_i$  obey the  $q$ -algebra discussed in appendix B. In this case, we can verify that the canonical distribution  $f(\vec{p})$  obtained from the *basic*-entropy (3.1) with the mean energy constraint

$$\int_{\mathcal{M}} H(\vec{p}) f(\vec{p}) d_q^{3N} x d_q^{3N} p = \langle \epsilon \rangle \quad (5.2)$$

can be factorized as follows:

$$f(\vec{p}) = \prod_{i=1}^N f(p_i). \quad (5.3)$$

Here,  $f(p_i)$  is the probability distribution function of a single particle

$$f(p_i) = \frac{1}{Z_{q,i}} E_q \left( -\beta_q \frac{p_i^2}{2m} \right), \quad (5.4)$$

with  $p_i^2 = p_{x,i}^2 + p_{y,i}^2 + p_{z,i}^2$  and

$$Z_{q,i} = \int_{\mathcal{M}} E_q \left( -\beta_q \frac{p_i^2}{2m} \right) d_q^3 x_i d_q^3 p_i = V_{i,q} \left( \frac{2\pi m_q}{\beta_q} \right)^{3/2}, \quad (5.5)$$

in units where  $\hbar = 1$ . In the above equation, we have posed  $V_{i,q} = \int_{\mathcal{M}} d_q^3 x_i$ , the fractal volume occupied by the  $i$ th particle and  $m_q = mA_q$  where

$$A_q = \frac{1}{\sqrt{\pi}} \int_0^\infty E_q(-x^2) d_q x \quad (5.6)$$

is a constant reducing to unity in the  $q \rightarrow 1$  limit. The canonical partition function of the whole system is given by

$$Z_q = \prod_{i=1}^N Z_{q,i} = V_q^N \left( \frac{2\pi m_q}{\beta_q} \right)^{3N/2}. \quad (5.7)$$

We may note that equation (5.4) can be interpreted as the  $q$ -deformed version of the Maxwell-Boltzmann distribution. It differs formally from the well-known classical distribution by the mere replacement of the standard exponential with its  $q$ -deformed generalization, consistently with the  $q$ -algebra underlying the mathematical structure of the theory.

Employing the distribution  $f(\vec{p})$ , we can compute the mean value of any observable associated with the system. In particular, the mean energy is given by

$$\langle \epsilon \rangle = q^{-\frac{3}{2}N} \left[ \frac{3}{2}N \right] \frac{1}{\beta_q}, \quad (5.8)$$

From equation (5.8), we recognize in the  $q \rightarrow 1$  limit the well-known result  $\langle \epsilon \rangle = \frac{3}{2}NT$ . We observe that

$$\langle \epsilon \rangle \rightarrow +\infty \quad \text{for } q \ll 0 \quad \text{and} \quad \langle \epsilon \rangle \rightarrow 0 \quad \text{for } q \gg 1, \quad (5.9)$$

which is a consequence of the dependence of  $f(\vec{p})$  on  $q$ . In fact, for smaller and smaller  $q$  the tail of the distribution is enhanced so that particles with high energy give a larger contribution. In contrast, for larger and larger  $q$  the cut-off inhibits the occupation by the system of the phase-space cells with high energy and only particles with lower and lower energy contribute to  $\langle \epsilon \rangle$ . In particular, for  $q \gg 1$  the distribution  $f(\vec{p}) \rightarrow \delta(\vec{0})$  so that only the fundamental level is populated. Similar arguments can be applied to justify the expression of the heat capacity given by

$$C_V = q^{-\frac{3}{2}N-1} \left[ \frac{3}{2}N \right], \tag{5.10}$$

which is a constant as in the undeformed classical case, but it is a monotonically decreasing function reducing to zero for  $q \gg 1$ .

It is important to outline that the free particle gas with the Hamiltonian (5.1) has the  $q$ -deformed particle distribution (5.4) only if we require, as a crucial assumption, that the particle momenta obey the  $q$ -algebra. On the other hand, if we do not employ the appropriate  $q$ -algebra, the same *free*  $q$ -deformed particle distribution (5.4) can be obtained by assuming the following  $N$ -body *interacting* Hamiltonian

$$H(\vec{p}) = -\frac{1}{\beta_q} \text{Ln}_q \left( \prod_{i=1}^N E_{1/q} \left( \beta_q \frac{p_i^2}{2m} \right) \right). \tag{5.11}$$

This suggests that the effects of the *basic*-deformation of a *free*-ideal gas can be viewed as an effective interaction described by the Hamiltonian (5.11). We remark that in the  $q \rightarrow 1$  limit the  $q$ -algebra reduces to the ordinary algebra used in the Hamiltonian (5.11). Furthermore, the same Hamiltonian (5.11) reduces to the Hamiltonian of an  $N$ -free particles system.

It may be important to clarify this point. In several papers [8, 46–48], it has been suggested that, starting from a deformed exponential derived through physically and/or mathematically justified arguments, it is possible to introduce a deformed sum in order to reproduce, in a deformed fashion, the well-known multiplicative rule of the standard exponential  $\exp(x + y) = \exp(x)\exp(y)$ . It has been conjectured (see for instance [49, 50]) that such a deformed sum can be employed, on physical grounds, to take into account the complex interactions arising among the many-body colliding particles of a nonlinear medium. This has been discussed, for instance, explicitly in the Tsallis-entropy framework [51]. In that case in fact, for the deformed sum of the energy values  $\mathcal{E}^A$  and  $\mathcal{E}^B$  belonging to two different subsystems A and B, it has been assumed that the expression describes the  $q$ -sum

$$\mathcal{E}^A \oplus_q \mathcal{E}^B = \mathcal{E}^A + \mathcal{E}^B + \frac{1-q}{\beta} \mathcal{E}^A \mathcal{E}^B, \tag{5.12}$$

and correspondingly the  $q$ -Boltzmann factor factorizes according to

$$\exp_q(-\beta(\mathcal{E}^A \oplus_q \mathcal{E}^B)) = \exp_q(-\beta\mathcal{E}^A) \exp_q(-\beta\mathcal{E}^B). \tag{5.13}$$

The same situation occurs in the Kaniadakis-entropy framework [7] where, by assuming

$$\mathcal{E}^A \oplus^\kappa \mathcal{E}^B = \mathcal{E}^A \sqrt{1 + (\kappa\beta\mathcal{E}^B)^2} + \mathcal{E}^B \sqrt{1 + (\kappa\beta\mathcal{E}^A)^2}, \tag{5.14}$$

for the energy levels, it has been shown that the  $\kappa$ -Boltzmann factor factorizes in

$$\exp_{\{\kappa\}}(-\beta(\mathcal{E}^A \oplus^\kappa \mathcal{E}^B)) = \exp_{\{\kappa\}}(-\beta\mathcal{E}^A) \exp_{\{\kappa\}}(-\beta\mathcal{E}^B). \tag{5.15}$$

Of course, all of this can also be reproduced within the formalism employed in the present work. In fact, we can verify that the following deformed sum

$$\mathcal{E}^A \oplus \mathcal{E}^B = -\frac{1}{\beta} \text{Ln}_q(E_{1/q}(\beta\mathcal{E}^A)E_{1/q}(\beta\mathcal{E}^B)), \tag{5.16}$$



which reduces to the ordinary sum in the  $q \rightarrow 1$  limit, fulfils the factorization rule

$$E_q(-\beta(\mathcal{E}^A \oplus \mathcal{E}^B)) = E_q(-\beta\mathcal{E}^A)E_q(-\beta\mathcal{E}^B), \quad (5.17)$$

for the *basic*-Boltzmann factor. We easily recognize in definition (5.16) the origin of the Hamiltonian (5.11). Although it is beyond the scope of the present paper, it is also possible to show that the *basic*-sum (5.16) obeys all the axiomatic properties where a well-defined sum must satisfy associativity, commutativity, existence and uniqueness of the inverse element and of the identity element. However, by applying the appropriate  $q$ -algebra introduced starting from the  $q$ -binomial expansion, the deformed sum  $x \oplus y$  is replaced, in a natural way, by the ordinary sum  $x + y$ . This aspect makes the *basic*-thermostatistics formalism very interesting, because the structure of the considered deformation implies a close and consistent realization in the well-known mathematical framework of  $q$ -calculus.

## 6. Conclusion

In this paper, we have studied a possible generalization of the thermostatistics theory of a classical system based on the  $q$ -deformed algebra. Starting from the definition of the *basic*-exponential, we have introduced a generalized entropic function and derived, by means of a consistent  $q$ -variational principle, a deformed probability distribution function which differs from the standard Gibbs distribution by the replacement of the ordinary exponential function with its generalization furnished by the *basic*-exponential. We have performed a preliminary investigation of some fundamental thermodynamic relations which are preserved consistently with the formalism of the  $q$ -calculus.

On physical grounds, it has been demonstrated that the distribution arising in this model exhibits a cut-off in the energy spectrum which is generally expected in those systems whose underlying dynamics is governed by long-range interactions. Such a feature has also been observed in other distributions proposed in the literature, obtained from generalized versions of the Boltzmann–Gibbs entropy. What is different here is the asymptotic behaviour of the distribution derived in this paper which does not match with the power-law behaviour typically shown by the other generalized distributions.

We have studied, within the present formalism, an  $N$ -body system of interacting particles described by the Hamiltonian (5.11), whose interaction vanishes in the  $q \rightarrow 1$  limit. By construction, the canonical distribution function of this system is formally equivalent to the one derived starting from the Hamiltonian (5.1), describing a system of non-interacting particles, where the momenta obeying the  $q$ -algebra originates from the  $q$ -binomial expansion (2.10). In this sense, the toy model described by the Hamiltonian (5.11) can be considered as the  $q$ -analogue of the ideal gas.

Jackson  $q$ -derivative and  $q$ -integral being the natural tools for describing discrete-scale invariance [22, 23], we expect that this study can be a very useful starting point on the *basic*-thermostatistic theory which can be strictly related to critical phenomena (such as growth processes, rupture, earthquake, financial crashes) with the existence of log-periodic oscillations deriving from a partial breakdown of the continuous scale-invariance symmetry into a discrete-scale-invariance symmetry, as occurs for instance in hierarchical lattice [52]. In fact, as mentioned in subsections 3.1 and 3.2, our study on the *basic*-thermostatistics incorporates implicitly a self-similarity in the parameter space, labelling the phase space of the system, and, consequently, a fractal structure in the energy spectrum emerges. In this context, it is remarkable to observe that, for example, the specific heat corresponding to systems with deterministic fractal energy is known to present log-periodic oscillations as a function of the temperature around a mean value given by a characteristic dimension of the

energy spectrum [53–55]. A detailed study of these critical phenomena in our formalism lies on the scope of this paper and will be the matter of future investigations.

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### Appendix A

As known, a possible way to obtain the stationary distribution of a system governed by a given entropy, under a set of suitable physically constraints, follows by means of the variational calculus on the constrained entropic form. In the case under investigation, accounting the  $q$ -algebra underlying our formalism, we have to deal with the following problem:

$$\delta\mathcal{F}(p) = 0, \quad (\text{A.1})$$

where, according to equation (3.3),

$$\mathcal{F}(p) = - \int_{\mathcal{M}} p(\lambda) \left[ \text{Ln}_q(p(\lambda)) + \sum_j \mu_j^* \phi_j(\lambda) \right] \mathrm{d}_q \lambda. \quad (\text{A.2})$$

Without loss of generality, we pose  $\mu_0^* = q^{-1}(1 + \mu_0) - 1$  and  $\mu_i^* = q^{-1}\mu_i$  for  $i = 1, \dots, M$ , where  $\mu_j$  are the Lagrange multipliers of the  $M + 1$  constraints (3.4) introduced in section 3.

By using the ansatz (3.5) in  $\mathcal{F}(p) \equiv \tilde{\mathcal{F}}(p(f))$ , equation (A.1) can be computed as follows:

$$\begin{aligned} \delta\tilde{\mathcal{F}}(f) &= \lim_{t \rightarrow 0} \left\{ \int_{\mathcal{M}} \left[ f(\lambda) + th(\lambda) - \sum_j \mu_j^* \phi_j(\lambda) \right] E_q(-f(\lambda) - th(\lambda)) \mathrm{d}_q \lambda \right. \\ &\quad \left. - \int_{\mathcal{M}} \left[ f(\lambda) - \sum_j \mu_j^* \phi_j(\lambda) \right] E_q(-f(\lambda)) \mathrm{d}_q \lambda \right\} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{M}} \left[ f(\lambda) + th(\lambda) - \sum_j \mu_j^* \phi_j(\lambda) \right] E_q(-f(\lambda) - th(\lambda)) \mathrm{d}_q \lambda \Big|_{t=0} \\ &= \int_{\mathcal{M}} \left[ h(\lambda) + \left( f(\lambda) + th(\lambda) - \sum_j \mu_j^* \phi_j(\lambda) \right) \frac{\frac{\mathrm{d}}{\mathrm{d}t} E_{1/q}(-th(\lambda))}{E_{1/q}(-th(\lambda))} \right] E_q(-f(\lambda)) \mathrm{d}_q \lambda, \end{aligned} \quad (\text{A.3})$$

where  $E_q(-f - th) = E_q(-f)E_{1/q}(-th)$  according to equation (2.16).

By taking into account definition (2.15) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{1/q}(-th(\lambda)) = \sum_{n=1} \frac{n}{[n]!} (-h(\lambda)) t^{n-1} \Big|_{t=0} = -h(\lambda), \quad (\text{A.4})$$

so that from equation (A.3) we obtain

$$\delta\tilde{\mathcal{F}}(f) = \int_{\mathcal{M}} h(\lambda) \left( 1 - f(\lambda) + \sum_j \mu_j^* \phi_j(\lambda) \right) E_q(-f(\lambda)) \mathrm{d}_q \lambda = 0. \quad (\text{A.5})$$

Accounting for the arbitrariness of the function  $h(\lambda)$  this last equation implies

$$f(\lambda) = 1 + \sum_j \mu_j^* \phi_j(\lambda), \quad (\text{A.6})$$

in accordance with equation (3.9) given in the text.

## Appendix B

We present a generalization of the algebra (2.10) to a trinomial with the purpose of extending the factorization formula of the *basic*-exponential.

First, let us briefly review the derivation of equation (2.16).

This can be shown easily by considering the Cauchy product among the series (2.15) and its analogue for  $q \rightarrow 1/q$ . We obtain

$$\begin{aligned} E_q(x)E_{1/q}(y) &= 1 + \left( \frac{x}{[1]!} + \frac{y}{[1]!} \right) + \left( \frac{x^2}{[2]!} + \frac{xy}{[1]![1]!} + \frac{y^2}{[2]!} \right) + \dots \\ &+ \left( \frac{x^n}{[n]!} + \frac{x^{n-1}y}{[n-1]![1]!} + \frac{qx^{n-2}y^2}{[n-2]![2]!} + \dots + \frac{q^{n(n-1)/2}y^n}{[n]!} \right) + \dots, \end{aligned} \quad (\text{B.1})$$

which, by means of equation (2.10), can be rewritten in the form

$$E_q(x)E_{1/q}(y) = 1 + \frac{(x+y)^{(1)}}{[1]!} + \frac{(x+y)^{(2)}}{[2]!} + \dots + \frac{(x+y)^{(n)}}{[n]!} + \dots, \quad (\text{B.2})$$

and coincides with the definition of  $E_q(x+y)$  given in equation (2.17).

To generalize this result, we consider the following  $q$ -binomial expansions:

$$\begin{aligned} (x+z)^{(0)} &= 1, \\ (x+z)^{(1)} &= x+z, \\ (x+z)^{(2)} &= x^2 + [2]xz + qz^2, \\ (x+z)^{(3)} &= x^3 + [3]x^2z + q[3]xz^2 + q^3z^3, \end{aligned} \quad (\text{B.3})$$

and so on. By redefining  $x \rightarrow x+y$  and consequently  $x^n \rightarrow (x+y)^{(n)}$ , we obtain

$$\begin{aligned} ((x+y)+z)^{(0)} &= 1, \\ ((x+y)+z)^{(1)} &= x+y+z, \\ ((x+y)+z)^{(2)} &= (x+y)^{(2)} + [2](x+y)^{(1)}z + qz^2 \\ &= x^2 + [2]xy + qy^2 + [2]xz + [2]yz + qz^2, \\ ((x+y)+z)^{(3)} &= (x+y)^{(3)} + [3](x+y)^{(2)}z + q[3](x+y)^{(1)}z^2 + q^3z^3 \\ &= x^3 + [3]x^2y + q[3]xy^2 + q^3y^3 + [3]x^2z \\ &+ [2]xyz + qy^2z + q[3]xz^2 + q[3]yz^2 + q^3z^3, \end{aligned} \quad (\text{B.4})$$

which implies the following factorization rule for the *basic*-exponential:

$$E_q(x+y+z) = E_q(x+y)E_{1/q}(z) = E_q(x)E_{1/q}(y)E_{1/q}(z). \quad (\text{B.5})$$

On the other hand, starting from the  $q$ -binomial expansion (B.3) and by posing  $z^n \rightarrow (y+z)_{1/q}^{(n)}$ <sup>5</sup> we form the  $q$ -trinomial expansion  $(x+(y+z)_{1/q})^{(n)}$  which implies the following decomposition:

$$E_q(x+y+z) = E_q(x)E_{1/q}(y+z) = E_q(x)E_{1/q}(y)E_q(z). \quad (\text{B.6})$$

<sup>5</sup> We have introduced the index  $1/q$  to indicate the replacement  $q \rightarrow 1/q$  in the expansion of the  $q$ -binomial given in equation (2.10).

Other possible factorization rules can be realized through the introduction of suitable  $q$ -deformed algebras, as can be seen by inspection. Extension to an arbitrary polynomial can also be easily obtained.

In the following, let us show the use of the above algebra in the derivation of the distributions (3.1.9), (3.2.7) and (3.3.7). Starting from the equality

$$x + y = 0, \quad (\text{B.7})$$

and employing the  $q$ -algebra (B.3), we can construct the  $q$ -binomial  $(x + y)^{(n)} = 0$  which of course vanishes for all  $n > 0$ . Consequently, according to definition (2.15), by dividing equation (B.7) by  $[n]!$  and summing up over  $n = 0, \dots, \infty$ , we obtain

$$E_q(x + y) = E_q(x)E_{1/q}(y) = 1. \quad (\text{B.8})$$

In particular, by applying this result to equation (3.1.8) which has the form (B.7), with  $x = \text{Ln}_q p(\epsilon)$  and  $y = q^{-1}(1 + \alpha)$ , we obtain

$$E_q(\text{Ln}_q p(\epsilon))E_{1/q}(q^{-1}(1 + \alpha)) = 1, \quad (\text{B.9})$$

so that

$$p(\epsilon) = E_q(-q^{-1}(1 + \alpha)), \quad (\text{B.10})$$

and according to the property (2.18) we obtain equation (3.1.9) given in section 3.1.

In the same manner, from the equality

$$\text{Ln}_q p(\epsilon) + q^{-1}(1 + \alpha + \beta\epsilon) = 0, \quad (\text{B.11})$$

given in equation (3.2.6), employing the  $q$ -algebra (B.4) with  $x = \text{Ln}_q p(\epsilon)$ ,  $y = q^{-1}(1 + \alpha)$  and  $z = q^{-1}\beta\epsilon$ , it follows that

$$E_q(\text{Ln}_q p(\epsilon))E_{1/q}(q^{-1}(1 + \alpha))E_{1/q}(q^{-1}\beta\epsilon) = 1, \quad (\text{B.12})$$

or equivalently

$$p_i = E_q(-q^{-1}(1 + \alpha))E_q(-q^{-1}\beta\epsilon), \quad (\text{B.13})$$

and again by using the property (2.18) it can be written in the form (3.2.7) given in section 3.2. Similar arguments can be applied to obtain the distribution (3.3.7) by employing the appropriate  $q$ -algebra with  $x = \text{Ln}_q p(\epsilon)$ ,  $y = q^{-1}(1 + \alpha)$ ,  $z = q^{-1}\beta\epsilon$  and  $u = q^{-1}\mu\beta N$ .

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